The Goldstone Theorem

wtorek, 5 listopada 2019 18:47

The Goldstone Theorem

Assumptions:
i) there is $\mathcal{L}$ symmetric under a continuous transformation G

ii) among other fields there are scalar fields ϕ_i, $i = 1 \dots n$ which are real (any complex fields can be always written as doubled number of real fields),

iii) the fields ϕ_i transform according to some (possibly reducible) representation of G

iv) for some $\phi_i \equiv v_i \neq 0$ $\quad \left. \frac{\partial V}{\partial \phi_i} \right|_{\phi_i = v_i} = 0$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi_i) + \dots$$

$$\phi_i \to \phi' = e^{-i\theta^a T_a} \phi \quad \Rightarrow \quad \phi_i \to \phi_i' = \phi_i - i\, \theta^a T^a_{ij} \phi_j + \dots$$

$$\text{symmetry} \Rightarrow \quad 0 = \delta V = \frac{\partial V}{\partial \phi_i} \underbrace{\delta \phi_i}_{-i\theta^a T^a_{ij} \phi_j} = -i \frac{\partial V}{\partial \phi_i} \theta^a (T^a)_{ij} \phi_j \quad \Big| \frac{\partial}{\partial \phi_k}$$

$$-i\theta^a T^a_{ij}\phi_j$$

$$0 = \frac{\partial^2 V}{\partial\phi_k \partial\phi_i}\,\theta^a T^a_{ij}\,\phi_j + \frac{\partial V}{\partial\phi_i}\,\theta^a T^a_{ij}\,\delta_{kj}$$

let's calculate the above at a minimum of $V(\phi)$:

$$\theta^a \underbrace{\frac{\partial^2 V}{\partial\phi_k \partial\phi_i}\bigg|_{\phi=v}}_{\equiv (M^2)_{ki}} (T^a v)_i = 0 \qquad \left(v = (v_1, \dots, v_n)^T \right)$$

$(M^2)_{ki}$ is a mass squared matrix, to see it let's expand the potential around its minimum

physical fields – fluctuations above the vacuum

$$V(\phi_i) = V(v) + \frac{\partial V}{\partial\phi_i}\bigg|_{\phi=v} (\phi - v)_i + \frac{1}{2}(\phi - v)_k \underbrace{\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\bigg|_{\phi=v}}_{\equiv (M^2)_{ki}} \overbrace{(\phi - v)_i} + \cdots$$

So we obtained

$$\theta^a (M^2)_{ki} (T^a v)_i = 0, \qquad (*)$$

what suggests that M^2 has eigenvectors with zero eigenvalues of the form $T^a v$.

what suggests that Π has eigenvectors with zero eigenvalues of the form $T v$.

Let's split generators into two groups: i) "unbroken by the vacuum": $T^a v = 0$

for those $(*)$ does not provide any information, $a = 1, \dots, m$,

these generators form an algebra and generate a subgroup of G, $H \subset G$

that leaves the vacuum invariant

ii) "broken by the vacuum": $T^a v \neq 0$

for $a = m+1, \dots, N$ (there are $N - m$ of them)

$\Downarrow$

each broken generator corresponds to zero mass particle (the Goldstone boson)

N is the number of the generators $\square$

In order to gauge away the Goldstone bosons let's parametrize ϕ_i by

$$\phi_i = \exp\left\{ i \sum^{N} T^c_{ij} \frac{\xi^c(x)}{\;} \right\} (v_j + \eta_j(x))$$

$$\phi_i = \exp\left\{ i \sum_{c=m+1}^{N} T^c_{ij} \frac{\xi^c(x)}{v} \right\} \left(v_j + \underbrace{\eta_j(x)}_{\text{remaining fluctuations of } \phi} \right)$$

$\sum_{c=m+1}^{N}$ sum over the broken generators

$$v \equiv \left(\sum v_j^2 \right)^{1/2}$$

Gauging away:

$$\phi(x) \longrightarrow U(x)\,\phi(x) \qquad \text{for} \qquad U(x) = \exp\left\{ -i \sum_{c=m+1}^{N} T^c \frac{\xi^c}{v} \right\}$$

$$A^a_\mu \longrightarrow A^{a\prime}_\mu$$

Mass matrix for the gauge bosons : $\frac{1}{2} A'^a_\mu A'^{b\mu} \underbrace{(T^a v)_i (T^b v)_i}_{\text{broken generators only}}$